

PACKING SUBGROUPS IN SOLVABLE GROUPS

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ABSTRACT. We show that any subgroup of a (virtually) nilpotent-by-polycyclic group satisfies the bounded packing property of Hruska-Wise([HW09]). In particular, the same is true about metabelian groups and linear solvable groups. However, we find an example of a finitely generated solvable group of derived length 3 which admits a finitely generated subgroup without the bounded packing property. In this example the subgroup is a metabelian retract also. Thus we obtain a negative answer to Problem 2.27 of [HW09]. On the other hand, we show that polycyclic subgroups of solvable groups satisfy the bounded packing property.

1. INTRODUCTION

Bounded packing was introduced by Hruska and Wise in [HW09]. It was motivated by the notion of width of subgroups due to Gitik, Mitra, Rips, and Sageev [GMRS97] and it was already known for quasiconvex subgroups of hyperbolic groups by the work of Mj(see Lemma 2.4, Lemma 3.3, Lemma 3.4 of [Mj08]) although Mj did not introduce the term ‘bounded packing’. However, Hruska and Wise reproved bounded packing for quasi-convex subgroups of hyperbolic groups in [HW09]. Their main theorem states that the relatively quasi-convex subgroups of relatively hyperbolic groups satisfy bounded packing under mild restrictions. Although bounded packing holds for all finitely generated subgroups of free groups, nilpotent groups and polycyclic groups ([Yan11]) it is not the case in general. For example, bounded packing for finitely generated subgroups of hyperbolic groups is false in general as is shown by Hruska-Wise. However, there are few examples where bounded packing is known to fail as is asserted by these authors. This paper is motivated by the following problem that these authors ask.

Problem (Question 2.13 of [HW09]): *Let S be a (countable) solvable group. Does every subgroup of S have bounded packing?*

The authors conjectured that the answer is *yes* for polycyclic groups. Later this was verified by W. Yang([Yan11]) who proved it more generally for any separable subgroup of a countable group. However, in this paper we give a negative answer for general solvable groups by producing a finitely generated solvable group of derived length 3 which admits a finitely generated subgroup without the bounded packing property. This subgroup is also a retract and it, therefore, answers Problem 2.27 of [HW09] negatively. However, we show that finitely generated subgroups of (virtually) metabelian groups have bounded packing. More generally, we show the following:

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Theorem 3.7 *Suppose we have an exact sequence of groups $1 \rightarrow N \rightarrow G \xrightarrow{\pi} Q \rightarrow 1$ where N is nilpotent and Q is polycyclic. Then any (finitely generated) subgroup of G satisfies the bounded packing property.*

We remark that metabelian groups are not subgroup separable in general. We mention a new set of examples of finitely generated groups admitting finitely generated subgroups without bounded packing using the work of Yves Cornuier [Cor06].

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2. BOUNDED PACKING PROPERTY

Definition 2.1. ([HW09]) *Suppose G is a countable group with a proper, left invariant metric d . Let $H \leq G$ be a subgroup. We say that H has bounded packing in G (with respect to d) if for any $D > 0$ there is a number $n = n(G, H, D)$ such that given any collection of distinct cosets $I = \{gH : g \in G\}$ of H in G such that $d(g_1H, g_2H) \leq D$ for all $g_1H, g_2H \in I$, we have $|I| \leq n$.*

By Lemma 2.2 of [HW09] we know that bounded packing for a subgroup of a countable group with a proper left invariant metric is independent of the particular metric. By the work of Higman, Neumann and Neumann ([HBN49]) we know that any countable group can be embedded in a 2-generated group; it follows that any countable group admits a left invariant proper metric and thus bounded packing of subgroups makes sense for all countable groups.

The following lemma is a summary of elementary properties in connection with bounded packing which are proved by Hruska-Wise in the section 2 of [HW09]. We will make repeated use of it later.

Lemma 2.2. ([HW09]) *Let G be a countable group with a proper left invariant metric. Then the following are true.*

- (1) *Every finite subgroup of G has bounded packing in G . Also every finite index subgroup of G has bounded packing in G .*
- (2) *Every normal subgroup of G has bounded packing in G .*
- (3) *For any sequence of subgroups $K \subset H \subset G$ if K has bounded packing in H and H has bounded packing in G , then K has bounded packing in G .*
- (4) *If H, K are two subgroup of G both of which have bounded packing in G then so does their intersection $H \cap K$.*
- (5) *Suppose we have a short exact sequence of groups $1 \rightarrow N \rightarrow G \xrightarrow{\pi} Q \rightarrow 1$. Let Q_1 be a subgroup of Q . Then Q_1 has bounded packing in Q if and only if $\pi^{-1}(Q_1)$ has the bounded packing in G .*

Theorem 2.3. (Theorem 2.12 of [HW09]) *If G is a virtually nilpotent group then each subgroup of N has bounded packing.*

Bounded Packing for Intersection of Subgroups

The following theorem was essentially proved by Yang in [Yan11]. To have it in this form requires a trivial modification of the proof of Yang [Yan11] for his theorem on bounded packing of separable subgroups. We include a proof for the

sake of completeness. However, although the statement is interesting we don't know of any remarkable application.

Theorem 2.4. ([Yan11]) *Suppose G is a countable group with a proper left invariant metric. Let $\{H_\alpha\}_{\alpha \in I}$ be any collection of subgroups of G such that each H_α has bounded packing in G . Let $K = \bigcap_{\alpha \in I} H_\alpha$. Then K has bounded packing in G .*

Proof : Let $D > 0$ be given and let $\{a_i K\}_{i \in \mathbb{N}}$ be any collection of left cosets of K such that $d(a_i K, a_j K) \leq D$ for all $i, j \in \mathbb{N}$. Then all these cosets are contained in a finite collection of double cosets $\{K, K b_1 K, K b_2 K, \dots, K b_n K\}$ where $d(1, b_i) \leq D$ and $b_i \notin K$ for all $1 \leq i \leq n$, for some $n \in \mathbb{N}$. Let $S := \{g \in G \setminus K : d(1, g) \leq D\}$. This is a finite set since the metric of G is proper. Note that $b_i \in S$ for all $1 \leq i \leq n$.

For each $t \in S$ choose $H_t := H_{\alpha_t}$, $\alpha_t \in I$ such that $t \notin H_{\alpha_t}$. Let $H = \bigcap_{t \in S} H_t$. We know that $K \subset H$. Note that H has bounded packing in G by Lemma 2.2(4).

We claim that all the distinct cosets of K from the above collection are contained in distinct cosets of H . Otherwise, suppose $a_i K, a_j K$ are distinct cosets of K and $a_i K \subset gH$, $a_j K \subset gH$ for some $g \in G$. This implies that $g^{-1}a_i K, g^{-1}a_j K \subset H$. This in turn implies that $K a_i^{-1} a_j K \subset H$. However, $K a_i^{-1} a_j K = K b_i K$ for some b_i whence $b_i \in H$. This is a contradiction since $b_i \in S$.

We thus have a collection of distinct cosets $\{a_i H\}$ of H such that $d(a_i H, a_j H) \leq D$ for all $i, j \in I$. Since H has bounded packing in G the set I is finite. This completes the proof. \square

Corollary 2.5. ([Yan11]) *Separable subgroups in any countable group satisfy the bounded packing property.*

Proof : We know that finite index subgroups have bounded packing by Lemma 2.2(1). Hence an intersection of any collection of finite index subgroups will have bounded packing by the above theorem. \square

Corollary 2.6. ([Yan11]) *Subgroups of polycyclic groups have bounded packing.*

Proof : We know that any subgroup of a polycyclic group is separable by a result of Hirsch ([Hir38]). \square

3. BOUNDED PACKING IN VIRTUALLY ABELIAN-BY-POLYCYCLIC GROUPS

In this section we prove that finitely generated subgroups of virtually abelian-by-polycyclic groups have bounded packing (Proposition 3.5). To prepare for the proof we need the following set of lemmas.

The proof of the following lemma is motivated by a similar result due to Cornuier (Lemma 3.18 of [Cor06]). Similar ideas are used throughout the paper, most prominently in Lemma 3.2.

Remark 3.1. (1) *To keep the notation simple, we will always denote the naturally defined quotient maps by π in all the proofs, unless otherwise needed.*
 (2) *Note that all the groups that we are concerned about sit in the middle of a short exact sequence. If instead we assume that the groups are “virtually” have this property then the conclusions will still hold by Lemma 2.2 (1) and (3). However, we omit the adverb “virtually” for simplicity.*

Lemma 3.2. *Suppose we have an exact sequence of groups $1 \rightarrow N \rightarrow G \xrightarrow{\pi} Q \rightarrow 1$ where N is abelian. Suppose $Q_1 := \pi(H)$ has bounded packing in Q . Let $K = H \cap N$*

and $H_1 := H/K$. Suppose H does not have bounded packing in G . Then there is a split exact sequence of the form

$$1 \rightarrow M \rightarrow G_1 \xrightarrow{\pi} H_1 \rightarrow 1$$

where M is a finitely generated submodule of the $\mathbb{Z}[H_1]$ -module N/K , and H_1 does not have bounded packing in G_1 .

Proof : By Lemma 2.2(5) $\pi^{-1}(Q_1)$ has bounded packing in G . Hence, by Lemma 2.2(3) H does not have bounded packing in $G' := \pi^{-1}(Q_1)$ since $H \subset G'$. Now going modulo K we get an exact sequence

$$1 \rightarrow N/K \rightarrow G'/K \xrightarrow{\pi} Q_1 \rightarrow 1.$$

By Lemma 2.2(5) $H_1 = H/K$ does not have bounded packing in G'/K . Therefore, there are infinitely many cosets $g_i H_1$ which are pairwise close where $g_i \in N/K$. It is clear that we can assume all the g_i 's to be in a finitely generated submodule, say M , of N/K using the properness of the metric on G'/K . Thus the bounded packing of H_1 fails in the subgroup, say G_1 , of G'/K generated by M and H_1 . Now we have a natural exact sequence $1 \rightarrow M \rightarrow G_1 \xrightarrow{\pi} Q_1 \rightarrow 1$. Note that H_1 surjects onto Q_1 and intersects M trivially. Hence π restricted to H_1 is an isomorphism onto Q_1 . Thus this is a split exact sequence and the action of H_1 (or equivalently Q_1) gives a finitely generated module structure on M . \square

Proposition 3.3. *Suppose a group G fits into an exact sequence of groups of the form $1 \rightarrow A \rightarrow G \rightarrow Q \rightarrow 1$ where A is a finitely generated abelian group and all (finitely generated) subgroups of Q satisfies the bounded packing property. Then all (finitely generated) subgroups of G satisfy the bounded packing property.*

Proof : Let H be any (finitely generated) subgroup of G . By the given conditions $\pi(H)$ has bounded packing in Q . Let $K = H \cap N$ and $H_1 = H/K$ as in the previous lemma. If H does not have bounded packing in G then by the above lemma there is a split exact sequence $1 \rightarrow M \rightarrow G_1 \xrightarrow{\pi} H_1 \rightarrow 1$ where M is a finitely generated $\mathbb{Z}[H_1]$ -module of N/K and H_1 does not have bounded packing in G_1 . In particular, M is a finitely generated abelian group.

Claim: H_1 is separable in G_1 .

Note that this gives a contradiction due to Corollary 2.5 and proves the lemma.

Proof of the claim: For all integer $k \geq 2$ $k.M$ is normal subgroup of G_1 and clearly the image of H_1 in $G/k.M$ has finite index since $M/k.M$ is a finite group. Therefore, the subgroup, say H_k , of G_1 generated by H_1 and $k.M$ is of finite index in G_1 . Therefore, it is enough to observe that $H_1 = \bigcap_{k \geq 2} H_k$. However, this is equivalent to showing $\bigcap_{k \geq 2} k.M = (0)$ which is clear since M is a finitely generated abelian group. \square

Remark 3.4. *We will see below (see Proposition 5.1) that the assumption of finite generation of A is necessary.*

Theorem 3.5. *Suppose we have an exact sequence of groups $1 \rightarrow N \rightarrow G \xrightarrow{\pi} Q \rightarrow 1$ where N is abelian and Q is polycyclic. Then any (finitely generated) subgroup of G satisfies the bounded packing property.*

Proof : Let $H \subset G$ be a subgroup. If $H \subset N$ then we are done by Lemma 2.2(2),(3) because N is abelian and normal in G . So we may suppose $\pi(H)$ is

nontrivial. By Corollary 2.6 $\pi(H)$ has bounded packing in Q , since Q is polycyclic. Since subgroup of a polycyclic group is again polycyclic $\pi(H)$ is polycyclic. Note that it is isomorphic to $H_1 = H/K$ where $K := N \cap H$. Now, if H does not have bounded packing in G then, by Lemma 3.2, we can find a finitely generated module M over H_1 such that H_1 does not have bounded packing in the semi-direct product $G_1 := H_1 \ltimes M$ where M is a finitely generated $R = \mathbb{Z}[H_1]$ -module.

Claim: H_1 is separable in G_1 .

First note that the claim gives a contradiction by Corollary 2.5 and proves the theorem. For the proof of the claim we make use of the following result of Roseblade([Ros76]) (and independently by Jategaonkar([Jat74])).

* *Finitely generated abelian-by-polycyclic groups are residually finite.*

Proof of the claim: By the result of Jategaonkar and Roseblade G_1 is residually finite. Therefore, G_1 has a plenty of finite index subgroups whose intersection is trivial. Intersecting finite index subgroups of G_1 with M we get finite index subgroups of M . Since G_1 is finitely generated there are only a finite number of finite index subgroups of a fixed index. Therefore, there is a sequence of finite index subgroups, say $\{M_k\}$ of M which are normal in G_1 . As in the proof of above proposition, we now go modulo M_k to get a split exact sequence

$$1 \rightarrow M/M_k \rightarrow G_1/M_k \rightarrow H_1 \rightarrow 1$$

Since M/M_k is finite, H_1 is of finite index in G_1/M_k . Thus the group generated by M_k and H_1 is of finite index in G_1 . Denote this group by H_k . Since G is residually finite it follows that $\bigcap M_k = (0)$. It follows then that $\bigcap H_k = H_1$. This means that H_1 is separable in G_1 . \square

Corollary 3.6. *Suppose a finitely generated group G fits into an exact sequence of groups of the form $1 \rightarrow A \rightarrow G \xrightarrow{\pi} Q \rightarrow 1$ where A is any abelian group and all (finitely generated) subgroups of Q satisfies the bounded packing property. Then all (finitely generated) subgroup H of G such that $\pi(H)$ is a polycyclic group, satisfies the bounded packing property.*

In particular, all polycyclic subgroups of G satisfies bounded packing.

Theorem 3.7. *Polycyclic subgroups of solvable groups satisfy bounded packing.*

Proof : This follows from Corollary 3.6 by induction on the derived length of solvable groups. \square

4. BOUNDED PACKING IN NILPOTENT-BY-POLYCYCLIC GROUPS

We are now ready to prove our main theorem.

Theorem 4.1. *Suppose a group G fits into an exact sequence of groups of the form $1 \rightarrow N \rightarrow G \xrightarrow{\pi} Q \rightarrow 1$ where N is any nilpotent group and Q is a polycyclic group. Then all subgroups of G satisfies the bounded packing property.*

Proof : The technique of proof for this theorem is similar to the one used for Lemma 3.2. Let H be a subgroup of G without bounded packing. We may assume that $\pi(H)$ is nontrivial by Theorem 2.3 and Lemma 2.2(2). We shall show a contradiction from this. As in the proof of Lemma 3.2 we reduce to the case $\pi(H) = Q$. Now, we shall use induction on the length of the central series of N . When N is abelian we are done by Theorem 3.5. Suppose N is a nonabelian

nilpotent group. Let C be its center. Then N/C has a strictly smaller central series than that of N .

Consider the short exact sequence

$$1 \rightarrow N/C \rightarrow G/C \xrightarrow{\pi} Q \rightarrow 1.$$

The image of H in G/C has bounded packing by induction. Call it \bar{H} . Hence H fails to have bounded packing in the inverse image of \bar{H} in G . Call it G_1 . Restricting the map π we have an exact sequence

$$1 \rightarrow K \rightarrow G_1 \xrightarrow{\pi} Q \rightarrow 1.$$

Now, G_1 is generated by C and H and K is a subgroup of N . Let $L = K \cap H$.

We observe that L is a normal subgroup of G_1 because L is normal in H and L is a subgroup of N which is normalized by C - the center of N . We are using here that G_1 is generated by H and C . Now, go modulo L to get the following exact sequence.

$$1 \rightarrow K/L \rightarrow G_1/L \xrightarrow{\pi} Q \rightarrow 1$$

Since H does not have bounded packing in G_1 , H/L can not have bounded packing in G_1/L by Lemma 2.2(5). However, note that $\frac{H}{L} \cap \frac{K}{L}$ is trivial and H/L surjects onto Q . Thus H/L is polycyclic. This means H/L has bounded packing in G_1/L by Theorem 3.7. This is a contradiction. \square

Corollary 4.2. *All subgroups of a linear solvable group satisfies the bounded packing property.*

By linear group we mean a subgroup of $GL(n, F)$ for some integer $n \in \mathbb{N}$ and a field F .

Proof : We know that all solvable linear groups are virtually nilpotent by abelian. Hence we can apply Remark 3.1(2) along with the above theorem. \square

Remark 4.3. *It is not true that all linear solvable groups are subgroup separable e.g. the solvable Baumslag-Solitar group $BS(1,2) = \langle a, b | aba^{-1} = b^2 \rangle$ is not subgroup separable, although there are a few instances where it is true, see e.g. [AF06].*

5. AN EXAMPLE

5.1. The example on solvable groups.

Proposition 5.1. *There is a finitely generated solvable group of derived length 3 which admits a finitely generated subgroup without the bounded packing property.*

For the proof of this proposition we need the following standard lemma.

Lemma 5.2. *Let H be a subgroup of a group G . Suppose H is acting on a vector space V over \mathbb{Q} . Then G acts in a natural way on $\oplus_{n \in S} V$ where S is a set of left coset representatives of H in G such that on the copy of V corresponding to $H \in S$ the H -action is the given one.*

Proof of the proposition: Let T be the group of positive rational numbers under multiplication. Then it acts naturally on the group $(\mathbb{Q}, +)$.

Note that T is isomorphic to the free abelian group of countably infinite rank. The set of primes forms a basis B . We can embed T in $Q := \mathbb{Z} \wr \mathbb{Z}$ as the obvious

normal subgroup with infinite cyclic quotient. Now using Lemma 5.2 we can obtain an action of this wreath product on $W := \bigoplus_{n \in \mathbb{Z}} \mathbb{Q}$.

Consider the semi-direct product $G := W \rtimes Q$. It is clear that W is a cyclic module over $\mathbb{Z}[Q]$ since \mathbb{Q} is a cyclic module over $\mathbb{Z}[T]$. Hence, G is a finitely generated group.

Claim: Q does not have bounded packing in G .

Proof of the claim: T is a normal subgroup of Q . Hence it suffices to show that T does not have bounded packing in G . Now, T is acting on W and leaves the 0-th copy of \mathbb{Q} invariant. We note that the cosets of T labelled by \mathbb{Q} are all close to each other, for suppose rT and sT are two distinct cosets where $r > s$. We need to find an element of the double coset $T(r-s)T$ which is uniformly close to the identity. Since $t := r-s > 0$, clearly we can choose $t.(r-s).t^{-1} \in T(r-s)T$. This gives us the element $1 \in \mathbb{Q}$. Since this does not depend on r, s we are done. \square

Remark 5.3. *Note that bounded packing for a general countable metabelian group is false since T does not have bounded packing in $T \rtimes \mathbb{Q}$.*

5.2. Some nonsolvable examples. Suppose G is a countable group with a left invariant proper metric and H is a subgroup of G . The for all $g_1, g_2 \in G$, $d_G(g_1H, g_2H) \leq D$ if and only if $d_G(Hg_1^{-1}g_2H, 1) \leq D$. Thus if H has only finitely many double cosets in G but infinite index in G then clearly H fails to have bounded packing. In other words, *if an infinite index subgroup of a group has finite bi-index (as defined in [Cor06]) then it fails to have bounded packing.*

Lemma 5.4. *Suppose N, Q are two finitely generated groups and Q acts 2-transitively on an infinite set X . Let $G = N \wr_X Q$ be the generalized restricted wreath product of Q with N . Then Q does not have bounded packing in G .*

Recall that the generalized wreath product here is the semi-direct product for the action of Q on $\bigoplus_{x \in X} N$ by permuting co-ordinates labelled by X . When $X = Q$ and the Q -action is by left multiplication, we get the usual restricted wreath product $N \wr Q$.

Proof : Fix $x \in N$ and denote by $x_t, t \in X$ the image of x in the copy of N corresponding to the co-ordinate $t \in X$. Then clearly x_tQ is an infinite collection of cosets of Q which are all pair-wise close to each other. \square

Corollary 5.5. *We have a split exact sequence of groups $1 \rightarrow N \rightarrow G \rightarrow \mathbb{Z} * \mathbb{Z}/2 \rightarrow 1$ where G is finitely generated such that $\mathbb{Z} * \mathbb{Z}/2$ does not have bounded packing in G .*

Proof : Note that $\mathbb{Z} * \mathbb{Z}/2$ acts naturally on $X := \mathbb{Z}$ where \mathbb{Z} acts by translation and $\mathbb{Z}/2$ exchanges 0 and 1 and keeps everything else fixed. This action is clearly 2-transitive. Hence, we can define G to be the generalized wreath product $\mathbb{Z} \wr_{\mathbb{Z}} \mathbb{Z} * \mathbb{Z}/2$. Then the above lemma finishes the proof.

Remark 5.6. *Several other examples of groups admitting 2-transitive actions on infinite sets are given by Cornulier (see Examples 3.4, 3.5, 3.6 in [Cor06]).*

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